A COMBINATORIAL APPROACH TO THE EXPONENTS OF MOORE SPACES

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Abstract. In this article, we give a combinatorial approach to the exponents of the Moore spaces. Our result states that the projection of the p^{r+1}-th power map of the loop space of the $(2n+1)$-dimensional mod p^r Moore space to its atomic piece containing the bottom cell $T^{2n+1}\{p^r\}$ is null homotopic for $n>1$, $p>3$ and $r>1$. This result strengthens the classical result that $\Omega T^{2n+1}\{p^r\}$ has an exponent p^{r+1}.

1. Introduction

The purpose of this article is to give a combinatorial approach to the exponents of Moore spaces. The exponent problem has been studied by various people with fruitful results [1, 3, 5, 10, 13, 14, 15, 20, 21] by using traditional methods. Our approach to the exponent of Moore spaces will be given by studying the combinatorics of the Cohen groups introduced in [2] together with collecting the minimal geometric information such as the classical Cohen-Moore-Neisendorfer decompositions and basic properties on the mod p^r homotopy groups of mod p^r Moore spaces [3, 4, 5].

Let us begin with a brief review on the Cohen groups. Let X be a pointed space. Recall that the James construction $J(X)$ is the free monoid generated by X subject to the single relation the basepoint $*\sim 1$, with weak topology. The James filtration $J_n(X)$ is given by the word length filtration of $J(X)$. Thus $J_n(X)$ is a quotient space of the n-fold Cartesian product $X^{\times n}$ as the coequalizer of the coordinate inclusions $d^i\colon X^{n-1}\to X^n$, $(y_1,\ldots,y_{n-1})\mapsto(y_1,\ldots,y_{i-1},*,y_i,\ldots,x_{y-1})$ for $1\leq i\leq n$. An important property of the James construction is that $J(X)$ is weakly homotopy equivalent to $\Omega\Sigma X$ if X is path-connected [9]. By using the James construction, one can get a combinatorial approach to the self-maps of loop suspensions in the following way. Let $F_n=\langle x_1,\ldots,x_n\rangle$ be the free group of rank n with a fixed choice of basis $x_1,\ldots,x_n$. Observe that the multiplication of $\Omega\Sigma X$ induces a group structure on $[X^{\times n},\Omega\Sigma X]$. Consider the naive representation

$$\tilde{e}_X\colon F_n\longrightarrow[X^{\times n},\Omega\Sigma X]$$

as a group homomorphism, which sends x_i to the homotopy class of the composite

$$X^{\times n}\xrightarrow{\pi_i}X\xhookrightarrow{E}\Omega\Sigma X,$$

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where π_i is the i-th coordinate projection and E is the canonical inclusion. It was discovered in [2] that for any co-H-space X,

$$\tilde{e}_X([[x_{i_1}, x_{i_2}], \ldots, x_{i_t}]) = 1$$

if $i_p = i_q$ for some $1 \le p < q \le t$. The group $K_n = K_n(x_1, \ldots, x_n)$ was then introduced as the quotient group of F_n subject to the above relations, with the property that $\tilde{e}_X$ induces a representation

$$e_X \colon K_n \longrightarrow [X^{\times n}, \Omega\Sigma X]$$

for any co-H-space X. In order to obtain self-maps of $\Omega\Sigma X$, the suspension splitting theorem of the James construction gives a good property that the quotient map $q_n \colon X^{\times n} \to J_n(X)$ induces a group monomorphism $q_n^* \colon [J_n(X), \Omega\Sigma X] \to [X^{\times n}, \Omega\Sigma X]$ and its image is given by the equalizer of the group homomorphisms $d^{i*} \colon [X^{\times n-1}, \Omega\Sigma X] \to [X^{\times n}, \Omega\Sigma X]$ for $1 \le i \le n$. Moreover

$$[\Omega\Sigma X, \Omega\Sigma X] \cong [J(X), \Omega\Sigma X] = \lim_n [J_n(X), \Omega\Sigma X]$$

is the inverse limit for any path-connected space X. The interpretation of d^{i*} in the Cohen group K_n is the projection homomorphism

$$d_i \colon K_n \longrightarrow K_{n-1}$$

with $d_i(x_j) = x_j$ for $j < i$, $d_i(x_i) = 1$ and $d_i(x_j) = x_{j-1}$ for $j > i$. Let H_n be the subgroup of K_n given as the equalizer of the group homomorphisms d_i for $1 \le i \le n$. The restriction of e_X on the subgroup H_n gives a representation

$$e_X \colon H_n \longrightarrow [J_n(X), \Omega\Sigma X]$$

for any co-H-space X. With taking inverse limit, let $H = \lim_n H_n$, one get a representation

$$e_X \colon H \longrightarrow [J(X), \Omega\Sigma X] \cong [\Omega\Sigma X, \Omega\Sigma X]$$

for any path-connected co-H-space X.

We should point out that the group K_n is isomorphic to Milnor's reduced free group introduced in his fundamental work on homotopy link theory [11]. A recent application of the group K_n in 4-manifolds is given in [6]. The importance of the Cohen groups K_n, H_n and H in homotopy theory is that H is a subgroup of the group of self natural transformations of the functor $\Omega\Sigma$ on path-connected co-H-spaces with its algebraic version through the Hurewicz homomorphism given exactly by the group of self natural transformations of the tensor algebra functor free abelian groups to coalgebras [16, 17, 22]. In particular, some fundamental objects in unstable homotopy theory, that is the Hopf invariants, the Whitehead product, the power maps and the loop of degree maps, are under controlled by the group H.

Suppose that the inclusion map $E \colon X \to \Omega\Sigma X$ has a finite order of p^r in the group $[X, \Omega\Sigma X]$. Then the representation $e_X \colon K_n \to [X^{\times n}, \Omega\Sigma X]$ factors through the group $K_n^{\mathbb{Z}/p^r} = K_n^{\mathbb{Z}/p^r}(x_1, \ldots, x_n)$, which is the quotient group of K_n by requiring $x_i^{p^r} = 1$ for $1 \le i \le n$. Similar to the integral version, the equalizer of the operations d_i on $K_n^{\mathbb{Z}/p^r}$ gives the subgroup $H_n^{\mathbb{Z}/p^r}$. The Cohen group $K_n^{\mathbb{Z}/p^r}$ services for the exponent problem, which is under exploration in this article. Observe that the particular element $\alpha_n = x_1 x_2 \cdots x_n \in H_n^{\mathbb{Z}/p^r} \le K_n^{\mathbb{Z}/p^r}$ has the geometric

interpretation as the homotopy class of the inclusion map $J_n(X) \to \Omega\Sigma X$. Suppose that $\alpha_n^{p^t} = 1$ in $K_n^{\mathbb{Z}/p^r}$. Then geometrically it means that the inclusion map $J_n(X) \to \Omega\Sigma X$ has an order bounded by p^t in the group $[J_n(X), \Omega\Sigma X]$. In particular the homotopy groups $\pi_*(\Omega\Sigma X) = \pi_{*+1}(\Sigma X)$ has an exponent bounded by p^t up to the range controlled by $J_n(X)$, namely below $(n+1)$ times the connectivity of X. When $n = 1$, $\alpha_1^{p^r} = 1$, which is the starting point. When n increases, the exponent of α_n also increases. For understanding the growth of α_n, it important and fundamental to understand the element $\alpha_n^{p^r}$ and the difference between $\alpha_{n+1}^{p^{r+1}}$ and $\alpha_n^{p^{r+1}}$. By using techniques in group theory, Lemma 2.6 gives a description of the element $\alpha_n^{p^r}$ and Proposition 2.7 gives a description of the difference between $\alpha_{n+1}^{p^{r+1}}$ and $\alpha_n^{p^{r+1}}$. Here, we should make a comment that the Stirling number appears naturally in this topic by Lemma 2.2.

It should be pointed out that, for any connected space X with a nontrivial reduced homology with coefficients in p-local integers, any power $p^t\colon \Omega\Sigma X \to \Omega\Sigma X$ is essential by [5, Theorem 3.10]. This property seems to discourage the study on the exponents of the single loop spaces. However, with taking the observation that $\Omega\Sigma X$ has various decompositions, one can ask the following question. Let T be the atomic retract of $\Omega\Sigma X$ containing the bottom cell. Is it possible that there is a choice of the projection map $\pi\colon \Omega\Sigma X \to T$ such that the composite

$$\Omega\Sigma X \xrightarrow{p^t} \Omega\Sigma X \xrightarrow{\pi} T$$

is null homotopic for some t?

By using combinatorial approach, we give the affirmed answer to the above question for Moore spaces. Our result is as follows. Recall [5, Corollary 1.9] that there is a homotopy decomposition

$$\Omega P^{2n+1}(p^r) \simeq T^{2n+1}\{p^r\} \times \Omega P(n, p^r)$$

for $p > 2$ and $n \geq 2$, where $P^m(p^r) = S^{m-1} \cup_{p^r} e^m$, the m-dimensional mod p^r Moore space, $P(n, p^r)$ is a wedge of mod p^r Moore spaces, and $T^{2n+1}\{p^r\}$ is the atomic retract of $\Omega P^{2n+1}(p^r)$.

Theorem 1.1. *There is a choice of the projection $\partial\colon \Omega P^{2n+1}(p^r) \to T^{2n+1}\{p^r\}$ such that composite*

$$\Omega P^{2n+1}(p^r) \xrightarrow{p^{r+1}} \Omega P^{2n+1}(p^r) \xrightarrow{\partial} T^{2n+1}\{p^r\}$$

is null homotopic for $p > 3$, $n > 1$ and $r > 1$.

This theorem strengthens the classical result [13] that $\Omega T^{2n+1}\{p^r\}$ has exponent p^{r+1} in the sense that $T^{2n+1}\{p^r\}$ already has exponent p^{r+1} in the above sense.

The article is organized as follows. In section 2, we explore the combinatorics of the Cohen groups. We give some remarks for potential applications for general spaces in section 3. In section 4, we give the applications to the Moore spaces. Theorem 1.1 is Theorem 4.1. In Section 5, we give the applications to the Anick spaces.

2. Combinatorics of the Cohen groups

In this section, p is an odd prime and $r \geq 1$. For elements $x, y, g_1, \dots, g_k$ of a group, we will use the standard commutator and left-normalized notation:

$$[x, y] := x^{-1}y^{-1}xy, \quad x^y := y^{-1}xy, \quad [g_1, \dots, g_k] := [[g_1, \dots, g_{k-1}], g_k].$$

For $i \geq 1$, we will use the following notation for the left-Engel brackets

$$[x,_1 y] := [x, y], \ [x,_i y] = [[x,_{i-1} y], y].$$

For $n \geq 1$, the Cohen group $K_n^{\mathbb{Z}/p^r} = K_n^{\mathbb{Z}/p^r}(x_1, \dots, x_n)$ is the quotient of a free group $F(x_1, \dots, x_n)$ of rank n by all left-normalized commutators

$$[x_{i_1}, \dots, x_{i_k}], \text{ such that } i_s = i_t \text{ for some } 1 \leq s, t \leq n, \ s \neq t$$

together with p^rth powers of generators $x_i^{p^r}$, $i = 1, \dots, n$. The group $K_n^{\mathbb{Z}/p^r}$ is nilpotent of class n.

In this paper, we will consider also the following subgroup of $K_n^{\mathbb{Z}/p^r}$. Let $\mathcal{B}_n$ be the subgroup of $K_n^{\mathbb{Z}/p}$ generated by all brackets

$$[x_{i_1}, \dots, x_{i_k}], \ k \neq p^t, \ t \geq 0.$$

For any configuration of brackets [[...], [[...]...]...], in a commutator of length k whose entrances are generators $\{x_1, \dots, x_n\}$ only, can be written as a product of left-normalized commutators of length k with generators as entrances. This follows from the definition of $K_n^{\mathbb{Z}/p^r}$ and the Hall-Witt identity. Therefore, any commutator of length $\neq p^t, t \geq 0$ whose entrances are generators, is in $\mathcal{B}_n$. Obviously, $\mathcal{B}_n$ is not normal in $K_n^{\mathbb{Z}/p}$.

The commutator calculus in groups $K_n^{\mathbb{Z}/p^r}$ are much simpler than in free nilpotent groups. We will need the following standard relations.

Lemma 2.1. *Let x be an element from the generating set $\{x_1, \dots, x_n\}$ and g any element of $K_n^{\mathbb{Z}/p^r}$. Then, for $k \geq 1$,*

$$[x, g^k] = \prod_{i=1}^{k} [x,_i g]^{\binom{k}{i}}; \tag{2.1}$$

$$(gx)^k = g^k x^k \prod_{i=1}^{k-1} [x,_i g]^{\binom{k}{i+1}}. \tag{2.2}$$

Proof. First we prove (2.1). For $k = 1$, this is obvious. Suppose that the formula is proved for a given k. Then, using the property of the group, that for all elements h_1, h_2, $[x, h_1]$ and $[x, h_2]$ commute, we get

$$[x, g^{k+1}] = [x, g][x, g^k]^g = [x, g][x, g^k][x, g^k, g] =$$
$$[x, g]^{k+1}[x,_{k+1} g] \prod_{i=2}^{k} [x,_i g]^{\binom{k}{i}+\binom{k}{i-1}} = \prod_{i=1}^{k+1} [x,_i g]^{\binom{k+1}{i}}.$$

The needed relation is proved.

To prove (2.2), we also use the induction on k. For $k = 1$ it is obvious. Suppose that (2.2) is proved for a given k. Then, using the relation $[x^k, g] = [x, g]^k$, we

obtain

$$(gx)^{k+1} = (gx)^k(gx) = g^k x^k (\prod_{i=1}^{k-1}[x,_i\, g])gx =$$
$$g^{k+1}x^{k+1}[x^k, g](\prod_{i=1}^{k-1}[x,_i\, g]^{\binom{k}{i+1}})\prod_{i=1}^{k-1}[x,_{i+1}\, g]^{\binom{k}{i+1}} =$$
$$g^{k+1}x^{k+1}[x,g]^{\binom{k+1}{2}}(\prod_{i=2}^{k-1}[x,_i\, g]^{\binom{k}{i+1}+\binom{k}{i}})[x,_k\, g] = g^{k+1}x^{k+1}\prod_{i=1}^{k}[x,_i\, g]^{\binom{k+1}{i+1}}.$$

The inductive step is done. □

For the convenience, we will work now in the group $K_{n+1}^{\mathbb{Z}/p^r} = K_{n+1}^{\mathbb{Z}/p^r}(x_1,\dots,x_{n+1})$. Observe that, for $l > n$,

$$[x_{n+1},_l\,(x_1\dots x_n)] = 1.$$

This follows from the simple observation that $K_{n+1}^{\mathbb{Z}/p^r}$ is nilpotent of class $n+1$. To describe the commutator $[x_{n+1},_l\,(x_1\dots x_n)]$ for $n \geq l$, we will need some special sets of permutations.

For a given $1 \leq l \leq n$, consider the set of permutations of $\{1,\dots,n\}$

$$\Sigma_l^n = \{(i_1,\dots,i_{k_1}, i_{k_1+1},\dots,i_{k_2},\dots,i_{k_{l-1}+1},\dots,i_{k_l}) \mid$$
$$i_{k_i+1} < \dots < i_{k_{i+1}},\ k_0 = 0,\ i = 1,\dots,l-1\}$$

That is, Σ_l^n consists of permutations on n letters such that they can be divided into l monotonic blocks. Some permutations can be divided into l monotonic blocks in different ways, for a permutation σ, the number of such divisions we denote by $d_l(\sigma)$. For example, here is the list of permutations from Σ_2^3 with values of d_2:

permutation	d_2
$(1,2,3)$	2
$(2,1,3)$	1
$(2,3,1)$	1
$(3,1,2)$	1
$(1,3,2)$	1
$(3,2,1)$	0

The following proposition follows immediately from the definition of the set Σ_l^n.

Lemma 2.2. *$\sum_{\sigma\in\Sigma_l^n} d_l(\sigma) = l!\left\{{n \atop l}\right\}$. Here $\left\{{n \atop l}\right\}$ is the second Stirling number.*

Indeed, the Stirling number $\left\{{n \atop l}\right\}$ is the number of ways to divide the set $\{1,\dots,n\}$ into l non-empty subsets. In each of l subsets we order the elements in the monotonic way. In this partition we can permute all l monotonic blocks. Each permutation σ appears in this way exactly $d_l(\sigma)$ times.

We will use later one more notation. For $1 \leq i \leq n$, denote

$$\Sigma_l^n(i) = \{(i_1,\dots,i_n) \in \Sigma_l^n \mid i_1 = i\}.$$

Lemma 2.3. *For any i, $\sum_{\sigma\in\Sigma_l^n(i)} d_l(\sigma)$ divides $(l-1)!$.*

Lemma 2.3 follows immediately from the definition of the set $\Sigma_l^n(i)$. If we consider some permutation from $\Sigma_l^n(i)$, we can fix the first monotonic block which starts with i and permute other $(l-1)$ monotonic blocks. One can easily prove explicit values of the above sum for some i-s. For example,

$$\sum_{\sigma\in\Sigma_l^n(1)} d_l(\sigma) = \left\{ {n \atop l} \right\}(l-1)!, \quad \sum_{\sigma\in\Sigma_l^n(n)} d_l(\sigma) = \left\{ {n-1 \atop l-1} \right\}(l-1)!$$

We will naturally extend the notation Σ_l^n for permutations on n (ordered) symbols, for example, for $N > n$, $\sigma \subset \{1, \dots, N\}$, we say that $\sigma \in \Sigma_l^n$ if it can be divided into l monotonic blocks. In a natural way, for these extended cases, one can define $d_l(\sigma)$.

Now we are able to describe the commutators $[x_{n+1},_l\, x_1 \dots x_n]$.

Lemma 2.4. *For any $l \geq 1$ and $n \geq l$,*

$$[x_{n+1},_l\, x_1 \dots x_n] = \prod_{i=l}^{n} \prod_{\sigma\in\Sigma_l^i, \sigma\subseteq\{1,\dots,n\}} [x_{n+1}, x_{\sigma(1)}, \dots, x_{\sigma(i)}]^{d_l(\sigma)}. \tag{2.3}$$

Proof. The proof is straightforward, by induction on l. For $l = 1$, we have

$$[x_{n+1}, x_1 \dots x_n] = \prod_{i=1}^{n} \prod_{j_1<\dots<j_i} [x_{n+1}, x_{j_1}, \dots, x_{j_i}].$$

The sets Σ_1^i have a single permutation $(1, \dots, i)$. In the notation used in the formulation of lemma, the product over such sets means exactly the product over ordered sets of i elements from $\{1, \dots, n\}$. That is, we have the needed formula for $l = 1$. Now assume that it is proved for a given l. We have

$$[x_{n+1},_{l+1}\, (x_1 \dots x_n)] = [[x_{n+1},_l\, (x_1 \dots x_n)], x_1 \dots x_n] =$$
$$\prod_{i=l}^{n} \prod_{\sigma\in\Sigma_l^i, \sigma\subseteq\{1,\dots,n\}} [[x_{n+1}, x_{\sigma(1)}, \dots, x_{\sigma(i)}], x_1 \dots x_n]^{d_l(\sigma)}.$$

For a fixed $\sigma \in \Sigma_l^i$ on letters $j_1, \dots, j_i$, consider the commutator

$$[x_{n+1}, x_{\sigma(1)}, \dots, x_{\sigma(i)}, x_1 \dots x_n].$$

Opening this commutator, we get

$$[x_{n+1}, x_{\sigma(1)}, \dots, x_{\sigma(i)}, x_1 \dots x_n] = \prod_{q_1<\dots<q_t} [x_{n+1}, x_{\sigma(1)}, \dots, x_{\sigma(i)}, x_{q_1}, \dots, x_{q_t}]. \tag{2.4}$$

We can assume that

$$\{q_1, \dots, q_t\} \cap \{j_1, \dots, j_i\} = \varnothing$$

Otherwise, the bracket is trivial. Observe that, the permutation

$$\{\sigma(1), \dots, \sigma(i), q_1, \dots, q_t\}$$

is from Σ_{l+1}^{i+t} on the set $\{j_1, \dots, j_i; q_1, \dots, q_t\}$, i.e. it is divided into $l+1$ monotonic blocks. The number $d_l(\sigma)$ is the number of divisions of $\{\sigma(1), \dots, \sigma(i), q_1, \dots, q_t\}$, which fixes the last monotonic block $(q_1, \dots, q_t)$. Observe that, the number of appearances of the bracket $[x_{n+1}, x_{\sigma(1)}, \dots, x_{\sigma(i)}, x_{q_1}, \dots, x_{q_t}]$ in the full product (2.4) is exactly $d_{l+1}(\{\sigma(1), \dots, \sigma(i), q_1, \dots, q_t\})$. The needed expression for the case $l+1$ follows. $\square$

Note that, one can present the product from (2.3) in terms of shuffles as follows

$$\prod_{\sigma\in\Sigma_l^i,\sigma\subseteq\{1,\dots,n\}}[x_{n+1},x_{\sigma(1)},\dots,x_{\sigma(i)}]^{d_l(\sigma)}=$$
$$\prod_{i_1+\dots+i_l=i,\ \sigma\in[i_1,\dots,i_l]-\text{shuffles}}[x_{n+1},x_{\sigma(1)},\dots,x_{\sigma(i)}].$$

Denote $K:=K_{n+1}^{\mathbb{Z}/p^r}$.

Lemma 2.5. *For $l\geq 2$,*

$$[x_{n+1},_l\,x_1\dots x_n]\in\gamma_2(K)^{(l-1)!}\gamma_2\gamma_2(K).$$

Proof. Denote $\tau_i(q)=\sum_{\sigma\in\Sigma_l^i(q)}d_l(\sigma)$. Since, modulo $\gamma_2\gamma_2(K)$, we can permute all letters in the brackets in (2.3) except first two, we have

$$\begin{aligned}(2.5)\quad &[x_{n+1},_l\,x_1\dots x_n]\equiv\\ &\prod_{i=l}^{n}\prod_{j_1<\dots<j_s<q<j_{s+1}<\dots<j_l}[x_{n+1},x_q,x_{j_1},\dots,x_{j_l}]^{\tau_i(s+1)}\mod\gamma_2\gamma_2(K)\end{aligned}$$

By lemma 2.3, all numbers $\tau_i(s+1)$ are divided by $(l-1)!$ and the result follows. $\square$

Lemma 2.6. *For any $n\geq 1$, and $r>1$, $(x_1\dots x_n)^{p^r}\in\gamma_2(K_n^{\mathbb{Z}/p^r})^{p^{r-1}}\gamma_2\gamma_2(K_n^{\mathbb{Z}/p^r})$.*

Proof. We prove by induction on n. For $n=1$, $x_1^{p^r}=1$. Assume that the needed property holds for a given n and prove it for $n+1$. By lemma 2.1,

$$\begin{aligned}(2.6)\quad (x_1\dots x_{n+1})^{p^r}&=(x_1\dots x_n)^{p^r}x_{n+1}^{p^r}\prod_i[x_{n+1},_{i-1}(x_1\dots x_n)]^{\binom{p^r}{i}}=\\ &(x_1\dots x_n)^{p^r}\prod_{p\mid i}[x_{n+1},_{i-1}(x_1\dots x_n)]^{\binom{p^r}{i}}.\end{aligned}$$

Using the equality (2.6), for the inductive step, it is enough to prove that

$$\prod_{p\mid i}[x_{n+1},_{i-1}(x_1\dots x_n)]^{\binom{p^r}{i}}\in\gamma_2(K)^{p^{r-1}}\gamma_2\gamma_2(K)$$

Given i, present it as $i=p^ze,\ (e,p)=1$. Moreover, we can assume that $z\geq 1$, since otherwise the whole bracket vanishes. It remains to show that

$$[x_{n+1},_{i-1}(x_1\dots x_n)]\in\gamma_2(K)^{p^{z-1}}\gamma_2\gamma_2(K).\tag{2.7}$$

This follows from lemma 2.5, since $(i-1)!$ is divisible by p^{z-1}. This proves (2.7) and finishes the inductive step. $\square$

For a subgroup H of K, we denote by $[x_{n+1},H]$ the subgroup of K, generated by elements $[x_{n+1},h],\ h\in H$.

Proposition 2.7. *For $n\geq 1$ and $r>1$,*

$$(x_1\dots x_{n+1})^{p^{r+1}}=(x_1\dots x_n)^{p^{r+1}}\gamma,$$

where

$$\gamma\in\gamma_2\gamma_2\gamma_2(K)[\gamma_2(K)^p,\gamma_2\gamma_2(K)](\gamma_2\gamma_2(K))^p\tag{2.8}$$

as well as

$$\gamma \in \mathcal{B}_{n+1}[\mathcal{B}_{n+1}, \gamma_2(K)^p][\mathcal{B}_{n+1}, \gamma_2\gamma_2(K)]. \tag{2.9}$$

Proof. [One of the key points of the proof of this proposition is the possibility to permute the elements from $[x_{n+1}, K]$. This possibility covers the problems which appear due to non-normality of the subgroup $\mathcal{B}_{n+1}$.]

It follows from (2.6) and the proof of the previous lemma that

$$(x_1 \dots x_{n+1})^{p^r} = (x_1 \dots x_n)^{p^r}\alpha,$$

where $\alpha \in [x_{n+1}, K_n^{\mathbb{Z}/p^r}]^{p^{r-1}}(\gamma_2\gamma_2(K_n^{\mathbb{Z}/p^r}) \cap [x_{n+1}, K])$. Taking the pth power of $(x_1 \dots x_n)^{p^r}\alpha$, we get

$$(x_1 \dots x_{n+1})^{p^{r+1}} = (x_1 \dots x_n)^{p^{r+1}}\alpha^p\beta,$$

where

$$\beta \in [[x_{n+1}, K]^{p^{r-1}}(\gamma_2\gamma_2(K_n^{\mathbb{Z}/p^r}) \cap [x_{n+1}, K]), \gamma_2(K_n^{\mathbb{Z}/p^r})^{p^{r-1}}\gamma_2\gamma_2(K_n^{\mathbb{Z}/p^r})]. \tag{2.10}$$

The needed element γ is $\alpha^p\beta$. Present α as $\alpha = \alpha_1\alpha_2$, where

$$\alpha_1 \in [x_{n+1}, K]^{p^{r-1}},$$
$$\alpha_2 \in (\gamma_2\gamma_2(K_n^{\mathbb{Z}/p^r}) \cap [x_{n+1}, K]).$$

The elements α_1 and α_2 commute, since they lie in $[x_{n+1}, K]$. Observe that $\alpha_1^p = 1$, since

$$[x_{n+1}, K]^{p^r} = 1.$$

For an element α_2, we have $\alpha_2 \in \gamma_2\gamma_2(K)$, therefore,

$$\alpha^p \in \gamma_2\gamma_2(K)^p\gamma_2\gamma_2\gamma_2(K).$$

Together with (2.10), we have a needed result (2.8).

Now we will prove (2.9). First consider the element α_2. It was already observed that $\alpha^p = \alpha_2^p$. The element α_2 is a product of elements of the form (and their inverses)

$$[[x_{i_1}, \dots, x_{i_t}], [x_{j_1}, \dots, x_{j_s}]],$$

where one of the generators in this brackets is x_{n+1}. If $t+s$ is not a power of p, then this bracket lies in $\mathcal{B}_{n+1}$, and we can move it to the term $\mathcal{B}_{n+1}$ in (2.9). If $t+s$ is a power of p, then one of t or s must not be a power of p, assume it is t. Then,

$$[[x_{i_1}, \dots, x_{i_t}], [x_{j_1}, \dots, x_{j_s}]]^p = [[x_{i_1}, \dots, x_{i_t}], [x_{j_1}, \dots, x_{j_s}]^p] \in [\mathcal{B}_{n+1}, \gamma_2(K)^p] \cap [x_{n+1}, K].$$

Now we consider the element β, which is a product of certain brackets from the subgroup $[x_{n+1}, K]$. These brackets (or their inverses) have one of the following forms:

(a) $[\beta_1, \beta_2]$, where β_1 and β_2 are of commutators in generators x_i-s, $\beta_1, \beta_2 \in \gamma_2\gamma_2(K)$;

(b) $[\beta_1, \beta_2]$, where $\beta_1 = \delta^{p^{r-1}}$, where δ is some commutator in generators and β_2 is some commutator in generators from $\gamma_2\gamma_2(K)$;

(c) $[\beta_1, \beta_2]$, where $\beta_i = \delta_i^{p^{r-1}}$, $i = 1, 2$ and δ_i are some commutators in generators.

For a commutator in generators ξ, denote by $|\xi|$ its commutator length, i.e. the number of the maximal term of the lower central series where ξ lies. Consider

the case (a). If $|\beta_1| + |\beta_2|$ is not a power of p, then the bracket $[\beta_1, \beta_2]$ lies in $\mathcal{B}_{n+1} \cap [x_{n+1}, K]$. Suppose that $|\beta_1| + |\beta_2|$ is a power of p. Then, one at least one of $|\beta_1|$ or $|\beta_2|$ is not a power of p, say β_1. Then $\beta_1 \in \mathcal{B}_{n+1}$ and, therefore, $[\beta_1, \beta_2] \in [\mathcal{B}_{n+1}, \gamma_2\gamma_2(K)]$. The same situation is in the case (b). If we assume that $|\beta_1|$ is not a power of p, we obtain an element from $[\mathcal{B}_{n+1}, \gamma_2\gamma_2(K)]$, if we assume that $|\beta_2|$ is not a power of p, we obtain an element from $[\mathcal{B}_{n+1}, \gamma_2(K)^{p^{r-1}}]$. In the same way we can handle the case (c). Observe also that, since $r > 1$, the case (c) becomes trivial, since

$$[\beta_1, \beta_2] = [\delta_1^{p^{r-1}}, \delta_2^{p^{r-1}}] = [\delta_1^{p^r}, \delta_2^{p^{r-2}}] = 1.$$

Since all brackets which we consider lie in $[x_{n+1}, K]$, we can permute them. This argument shows that the element γ satisfies the needed property (2.9). ☐

Corollary 2.8. *For any $n \geq 1$ and $r > 1$,*

$$(x_1 \dots x_n)^{p^{r+1}} \in \gamma_2\gamma_2\gamma_2(K_n^{\mathbb{Z}/p^r})[\gamma_2(K_n^{\mathbb{Z}/p^r})^p, \gamma_2\gamma_2(K_n^{\mathbb{Z}/p^r})](\gamma_2\gamma_2(K_n^{\mathbb{Z}/p^r}))^p$$

Observe that, for $r = 1$, the situation is different. In this case,

$$(x_1 \dots x_n)^{p^3} \in \gamma_2\gamma_2\gamma_2(K_n^{\mathbb{Z}/p}),$$

what can be easily proved by induction on n.

3. The geometric candidates for the subgroup $\mathcal{B}_n$ of K_n

The candidates from the subgroup $\mathcal{B}_n$ of K_n can be obtained from functorial decompositions of the loop-suspension functor on path-connected p-local co-H-spaces. Let us recall some results from [16, 17]. Let V be a module over the field $\mathbb{Z}/p$. The tensor algebra $T(V)$ is a Hopf algebra by saying V primitive. With forgetting the algebra structure, we have the functor T from modules to coalgebras. According to [16], there are functors $B^{\max}$ and $A^{\min}$ from modules to coalgebras with the properties

1) $A^{\min}$ is an indecomposable functor from modules to coalgebras;
2) there is a functorial coalgebra isomorphism

$$T(V) \cong B^{\max}(V) \otimes A^{\min}(V) \tag{3.1}$$

with $V \subseteq A^{\min}(V)$.

Here $B^{\max}(V)$ can be chosen a functorial sub Hopf algebra of $T(V)$ with a left functorial coalgebra inverse. According to [17, Section 2], the functorial coalgebra decomposition (3.1) holds over p-local integers. From this, [16, Theorem 1.5] can be extended over p-local integers and so we have an important property on the Lie powers of tensor length n

$$L_n(V) \subseteq B^{\max}(V) \text{ if } n \text{ is not a power of } p \tag{3.2}$$

for any free module V over p-local integers. (**Note.** Property (3.2) holds for any choice of the functor $B^{\max}$.)

The algebraic functors $A^{\min}$ and $B^{\max}$ admits geometric realization in the sense of [16, 17] that there are homotopy functors $A^{\min}$ and $Q^{\max}$ from path-connected p-local co-H-spaces to spaces with the following properties

1) $Q_n^{\max}(X)$ is a functorial retract of $\Sigma X^{\wedge n}$.

2) There is a functorial fibre sequence

$$A^{\min}(X) \xrightarrow{j_X} \bigvee_{n=2}^{\infty} Q_n^{\max}(X) \xrightarrow{\pi_X} \Sigma X$$

with $j_X \simeq *$. Here, the map π_X is given as a composite

$$\pi_X \colon Q_n^{\max}(X) \hookrightarrow \Sigma X^{\wedge n} \xrightarrow{W_n} \Sigma X, \tag{3.3}$$

where W_n is the Whitehead product.

3) There is a functorial decomposition

$$\Omega\Sigma X \simeq A^{\min}(X) \times \Omega\Big(\bigvee_{n=2}^{\infty} Q_n^{\max}(X)\Big).$$

4) Let $B^{\max}(X) = \Omega(\bigvee_{n=2}^{\infty} Q_n^{\max}(X))$. Then the mod p homology

$$H_*(A^{\min}(X)) \cong A^{\min}(\tilde{H}_*(X)) \text{ and } H_*(B^{\max}(X)) \cong B^{\max}(\tilde{H}_*(X)).$$

(**Note.** The geometric functors $A^{\min}$ and $B^{\max}$ can be generalized for decomposing any looped co-H-spaces [18, 19]. Here we are only interested in the case that $A^{\min}(X)$ and $B^{\max}(X)$ for co-H-spaces X.)

Proposition 3.1. *Let X be any path-connected p-local co-H-space. Then there is a homotopy commutative diagram*

$$\begin{tikzcd} B^{\max}(X) \arrow[r, "\Omega\pi_X"] & \Omega\Sigma X \\ X^{\wedge n} \arrow[u, dashed] \arrow[ur, "S_n"'] & \end{tikzcd}$$

for n not a power of p, where S_n is the n-fold Samelson product.

Proof. The assertion follows from Property (3.2) and the following commutative diagram

$$\begin{tikzcd} {[X^{\times n}, B^{\max}(X)]} \arrow[d, hook] & \mathcal{B}_n^{\max} \arrow[l] \arrow[r, "\cong"] \arrow[d, hook] \arrow[dr, phantom, "\text{pull-back}" near start] & \mathrm{coalg}(C(\ -\ )^{\otimes n}, B^{\max}(\ -\ )) \arrow[d, hook] \\ {[X^{\times n}, \Omega\Sigma X]} & K_n^{\mathbb{Z}_{(p)}} \arrow[l, "e_X"'] \arrow[r, "\cong"] & \mathrm{coalg}(C(\ -\ )^{\otimes n}, T(\ -\ ), \end{tikzcd}$$

where $C(V) = V \oplus \mathbb{Z}_{(p)}$ with trivial comultiplication as a functor from free $Z_{(p)}$-modules to coalgebras, the terms in the right column means the group of natural coalgebra transformations, $K_n^{\mathbb{Z}_{(p)}} = K_n^{\mathbb{Z}_{(p)}}(x_1, \ldots, x_n)$ is the Cohen group over p-local integers [22, Section 1.4] and e_X is the representation of the Cohen group on $[X^{\times n}, \Omega\Sigma X)]$ which sends x_i to the homotopy class of the composite

$$X^{\times n} \xrightarrow{i\text{−th coordinate projection}} X \hookrightarrow \Omega\Sigma X.$$

$\square$

The groups $\mathcal{B}_n^{\max}$ defined as above are the candidates for the subgroup $\mathcal{B}_n$ of the Cohen group K_n over $\mathbb{Z}_{(p)}$ or $\mathbb{Z}/p^r$ with the desired property that any commutator of length $\neq p^t, t \geq 0$ whose entrances are generators, is in $\mathcal{B}_n$. For a given co-H-space X, the $B^{\max}(X)$ can be a starting candidate for producing the subgroups $\mathcal{B}_n$. The derived series discussed in section 2 occurs naturally for resolutions of co-H-spaces by fibrations into H-spaces with the following observation. Let Y be an H-space. Let $f\colon \Sigma X \to Y$ be a map with a fibre sequence $F_f \xrightarrow{j} \Sigma X \xrightarrow{f} Y$, where F_f is the homotopy fibre of f. Then $\gamma_2([Z, \Omega\Sigma X]) \leq \mathrm{Im}(\Omega j_*\colon [Z, \Omega F_f] \to [Z, \Omega\Sigma X]$ for any spaces Z. By taking another map f_1 from F_f to an H-space Y_1 with the homotopy fibre F_{f_1}, then $\gamma_2\gamma_2([Z, \Omega\Sigma X])$ lies in the image from $[Z, \Omega F_1]$. Since the H-space resolutions for co-H-spaces seem out of control under current technology, we concentrate on the discussions on Moore spaces for highlighting the ideas of combinatorial approach in homotopy theory in next sections.

4. Applications to the Moore Spaces

Let us consider the Moore space $P^{2n+1}(p^r)$ with $n > 1$ and $p > 3$. The hypothesis $n > 1$ is used so that $P^{2n}(p^r)$ is a co-H-space, and the hypothesis $p > 3$ is used so that the mod p^r homotopy groups $\pi_*(\Omega P^{2n+1}(p^r); \mathbb{Z}/p^r)$ is a Lie algebra [3, Proposition 6.2]. Recall from [5], there is a fibre sequence

$$T^{2n+1}\{p^r\} \xrightarrow{j} P(n, p^r) \xrightarrow{\tilde{\pi}} P^{2n+1}(p^r). \tag{4.1}$$

where $T^{2n+1}\{p^r\}$ is the atomic piece of $\Omega P^{2n+1}(p^r)$ containing the bottom cell for $n > 1$, the map j is null homotopic, $P(n, p^r)$ is a wedge of mod p^r Moore spaces given as a retract of $\bigvee_{k=2}^{\infty} \Sigma(P^{2n}(p^r))^{\wedge k}$ and the map $\tilde{\pi}$ is given as a composite

$$\tilde{\pi}\colon P(n, p^r) \hookrightarrow \bigvee_{k=2}^{\infty} \Sigma(P^{2n}(p^r))^{\wedge k} \xrightarrow{\bigvee_{k=2}^{\infty} W_k} P^{2n+1}(p^r) \tag{4.2}$$

with W_k the iterated Whitehead product. Let $\partial\colon \Omega P^{2n+1}(p^r) \to T^{2n+1}\{p^r\}$ be the connecting map of the fibre sequence (4.1). Since j is null homotopic, the map

$$\psi = (\partial, \tilde{\pi})\colon \Omega P^{2n+1}(p^r) \simeq T^{2n+1}\{p^r\} \times \Omega P(n, p^r)$$

is a homotopy equivalence.

Theorem 4.1. *The composite*

$$\Omega P^{2n+1}(p^r) \xrightarrow{p^{r+1}} \Omega P^{2n+1}(p^r) \xrightarrow{\partial} T^{2n+1}\{p^r\}$$

is null homotopic for $p > 3$, $n > 1$ and $r > 1$.

Some preliminary settings are required before we prove this theorem. Recall that the mod p homology $H_*(\Omega P^{2n+1}(p^r)) = T(V)$ as a Hopf algebra, where $V = \tilde{H}_*(P^{2k}(p^r))$ having a basis $\{u, v\}$ with $|v| = 2n$, $|u| = 2n - 1$ and the r-th Bockstein $\beta^r v = u$. Under the hypothesis that $n > 1$, $H_*(\Omega P^{2n+1}(p^r)) = T(u, v)$ is a primitively generated Hopf algebra. In any Lie algebra L with $x, y \in L$, $\mathrm{ad}^0(y)(x) = x$ and $\mathrm{ad}^k(y)(x) = [x, \mathrm{ad}^{k-1}(y)(x)]$ for $k \geq 1$. Let

$$\tau_k = \mathrm{ad}^{p^k-1}(v)(u) \text{ and } \sigma_k = \sum_{j=1}^{p^k-1} \frac{1}{2p}\binom{p^k}{j}[\mathrm{ad}^j(v)(u), \mathrm{ad}^{p^k-j}(v)(u)].$$

By [5], the mod p homology $H_*(T^{2n+1}\{p^r\})$ is isomorphic to the free graded commutative algebra generated by u, v, τ_k, σ_k with $k \geq 1$ as a graded coalgebra. Let $L(V) \subseteq T(V)$ be the free graded Lie algebra generated by V. From the fibre sequence

$$\Omega P(n, p^r) \xrightarrow{\Omega\tilde{\pi}} \Omega P^{2n+1}(p^r) \xrightarrow{\partial} T^{2n+1}\{p^r\},$$

the sub Lie algebra

$$L(P(n, p^r)) = L(V) \cap \mathrm{Im}(\Omega\tilde{\pi}_* \colon H_*(\Omega P(n, p^r)) \to H_*(\Omega P^{2n+1}(p^r)))$$

can be described by the following diagram

$$\begin{tikzcd}
L(P(n,p^r)) \arrow[d, hook] \arrow[dr, hook] & & \\
{[L(V), L(V)]} \arrow[r, hook] \arrow[d, two heads] & L(V) \arrow[r, two heads] & L(V)^{\mathrm{ab}} \\
\sum\limits_{k=1}^{\infty} L(\tau_k, \sigma_k)^{\mathrm{ab}}, & &
\end{tikzcd} \tag{4.3}$$

where the row and the column are short exact sequences of graded Lie algebras and $\sum\limits_{k=1}^{\infty} L(\tau_k, \sigma_k)^{\mathrm{ab}}$ is the product of the abelian graded Lie algebras. The mod p homology

$$\tilde{H}_*(P(n, p^r)) \cong \Sigma L(P)^{\mathrm{ab}},$$

the suspension of the module $L(P)^{\mathrm{ab}}$, and

$$H_*(\Omega P(n, p^r)) \cong U(L(P)) \cong T(L(P)^{\mathrm{ab}},$$

where $U(L)$ is the universal enveloping algebra of a Lie algebra L.

Let $K_k(P)$ be the subgroup of $[(P^{2n}(p^r))^{\times k}, \Omega P^{2n+1}(p^r)]$ generated by the homotopy classes of the composites

$$x_i(P) \colon (P^{2n}(p^r))^{\times k} \xrightarrow{\pi_i} P^{2n}(p^r) \hookrightarrow \Omega P^{2n+1}(p^r),$$

where π_i is the i-th coordinate projection. Let

(4.4)

$$\mathcal{B}_k(P) = K_q(P) \cap \mathrm{Im}(\Omega\tilde{\pi}_* \colon [(P^{2n}(p^r))^{\times k}, \Omega P(n, p^r)] \to [(P^{2n}(p^r))^{\times k}, \Omega P^{2n+1}(p^r)]).$$

Lemma 4.2. *With the notations as above, $\gamma_2(\gamma_2(K_k(P)) \leq \mathcal{B}_k(P)$ for each $k \geq 2$.*

Proof. Recall that if $f \colon P^s(p^r) \to \Omega X$ and $g \colon P^t(p^r) \to \Omega X$. According to [14, (5.8) and (5.9)], the usual Samelson product $[f, g] \colon P^s(p^r) \wedge P^t(p^r) \to \Omega X$ decomposes as two maps

$$[f, g] \colon P^{s+t}(p^r) \to P^{s+t}(p^r) \vee P^{s+t-1}(p^r) \simeq P^s(p^r) \wedge P^t(p^r) \xrightarrow{[f,g]} \Omega X, \tag{4.5}$$

which is called the mod p^r Samelson product, and

$$\{f, g\} \colon P^{s+t-1}(p^r) \to P^{s+t-1}(p^r) \vee P^{s+t-1}(p^r) \simeq P^s(p^r) \wedge P^t(p^r) \xrightarrow{[f,g]} \Omega X \tag{4.6}$$

with $\{f,g\}=[\beta^r f,g]+(-1)^{a+1}[f,\beta^r g]$, where β^r is the Bockstein operation in the sense of [12]. Observe that the mod p^r homology $H_*(\Omega P^{2n+1};\mathbb{Z}/p^r)$ is a free $\mathbb{Z}/p$-module with $H_*(\Omega P^{2n+1};\mathbb{Z}/p^r)=T(u_r,v_r)$ as a Hopf algebra with $|u_r|=2n-1$ and $|v_r|=2n$. Following [3], let $\mu\in\pi_{2n-1}(\Omega P^{2n+1};\mathbb{Z}/p^r)$ and $\nu\in\pi_{2n}(\Omega P^{2n+1};\mathbb{Z}/p^r)$ be the elements in mod p^r homotopy groups whose Hurewicz image are given by u_r and v_r, respectively. Since the Hurewicz homomorphism

$$H\colon \pi_*(\Omega P^{2n+1}(p^r);\mathbb{Z}/p^r)\longrightarrow H_*(\Omega P^{2n+1}(p^r);\mathbb{Z}/p^r)$$

is a morphism of graded Lie algebras, the sub Lie algebra of $\pi_*(\Omega P^{2n+1}(p^r);\mathbb{Z}/p^r)$ generated by μ,ν is a free Lie algebra $L(\mu,\nu)$, which embeds into mod p^r homology under the Hurewicz homomorphism. By formulae (4.5) and (4.6), the iterated Samelson product

$$S_t\colon (P^{2n}(p^r))^{\wedge t}\longrightarrow \Omega P^{2n+1}(p^r)$$

decomposes as a linear combination of Lie elements in $L(\mu,\nu)$ for $t\geq 1$. Let

$$\tilde{\pi}'\colon \Sigma^{-1}P(n,p^r)\longrightarrow \Omega P^{2n+1}(p^r)$$

be the adjoint map of $\tilde{\pi}$. By definition (4.2), the homotopy class of the map $\tilde{\pi}'$ restricted to each factor of mod p^r Moore spaces in $\Sigma^{-1}P(n,p^r)$ is given by an element in $L(\mu,\nu)$. Let $\tilde{L}(P(n,p^r))$ be the sub Lie algebra of $\pi_*(\Omega P^{2n+1}(p^r))$ generated by the homotopy classes of the map $\tilde{\pi}'$ restricted to each factor of mod p^r Moore spaces in $\Sigma^{-1}P(n,p^r)$. Then

$$\tilde{L}(P(n,p^r))\subseteq \mathrm{Im}(\Omega\tilde{\pi}_*\colon \pi_*(\Omega P(n,p^r);\mathbb{Z}/p^r)\to\pi_*(\Omega P^{2n+1}(p^r);\mathbb{Z}/p^r)). \tag{4.7}$$

By using the property that the Hurewicz homomorphism to the mod p^r homology restricted to $L(\mu,\nu)$ is injective, sub Lie algebra $\tilde{L}(P(n,p^r))$ of $L(\mu,\nu)$ can be described by diagram (4.3) with that $L(V)$ is replaced by $L(\mu,\nu)$, $L(P(n,p^r))$ is replaced by $\tilde{L}(P(n,p^r))$, and τ_k,σ_k are replaced by their corresponding Lie elements in $L(\mu,\nu)$. It follows that

$$[[L(\mu,\nu),L(\mu,\nu)],[L(\mu,\nu),L(\mu,\nu)]]\leq \tilde{L}(P(n,p^r)) \tag{4.8}$$

Observe that the subgroup $\gamma_2(\gamma_2(K_k(P)))$ is generated by the commutators

$$x_{I,J}=[[[x_{i_1}(P),x_{i_2}(P)],\ldots,x_{i_s}(P)],[[x_{j_1}(P),x_{j_2}(P)],\ldots,x_{j_t}(P)]$$

for $1\leq i_1,\ldots,i_s,j_1,\ldots,j_t\leq k$. Note that the geometric interpretation of the commutator $[[x_{i_1}(P),x_{i_2}(P)],\ldots,x_{i_s}(P)]$ is the homotopy class of the composite

$$(P^{2n}(p^r))^{\times q}\xrightarrow{\pi_I}(P^{2n}(p^r))^{\wedge s}\xrightarrow{S_s}\Omega P^{2n+1}(p^r),$$

where π_I is given as a composite of a coordinate projection $(P^{2n}(p^r))^{\times k}\to(P^{2n}(p^r))^{\times s}$ followed by the pinch map $(P^{2n}(p^r))^{\times s}\to(P^{2n}(p^r))^{\wedge s}$. By using the property that S_s decomposes as a linear combination of Lie elements in $L(\mu,\nu)$ together with properties (4.7) and (4.8), we have

$$x_{I,J}\in\mathrm{Im}(\Omega\tilde{\pi}_*\colon [(P^{2n}(p^r)^{\times k},\Omega P(n,p^r)]\to[(P^{2n}(p^r)^{\times k},\Omega P^{2n+1}(p^r)]).$$

The assertion follows. $\square$

Proof of Theorem 4.1. Let $J(X)$ be the James construction on a pointed space X with the James filtration $J_k(X)$. Let $q_k\colon X^{\times k}\to J_k(X)$ be the projection map and let

$$d^i\colon X^{\times k-1}\to X^{\times k},\quad (x_1,\ldots,x_{k-1})\mapsto(x_1,\ldots,x_{i-1},*,x_i,\ldots,x_{k-1})$$

be the coordinate inclusion for $1 \le i \le k$. Let $\mathcal{H}_k(X, \Omega Y)$ be the equalizer of the group homomorphisms

$$d^{i*} \colon [X^{\times k}, \Omega Y] \longrightarrow [X^{\times k-1}, \Omega Y]$$

for $1 \le i \le k$. By [22, Theorem 1.1.5], $q_k^* \colon [J_k(X), \Omega Y] \to [X^{\times k}, \Omega Y]$ is a group monomorphism with its image given by $\mathcal{H}_k(X, \Omega Y)$. Moreover the inclusion $J_{k-1}(X) \to J_k(X)$ induces a group epimorphism $[J_k(X), \Omega Y] \twoheadrightarrow [J_{k-1}(X), \Omega Y]$ with

$$[J(X), \Omega Y] \cong \lim_k [J_k(X), \Omega Y] \cong \lim_k \mathcal{H}_k(X, \Omega Y)$$

being given by the inverse limit. We identify the group $[J_k(X), \Omega Y]$ with its image in $[X^{\times k}, \Omega Y]$ under group monomorphism q_k^* and the group $[J(X), \Omega Y]$ with the inverse limit $\mathcal{H}(X, \Omega Y) = \lim\limits_k \mathcal{H}_k(X, \Omega Y)$.

For any pointed space X, we identify the group $[X, \Omega P(n, p^r)]$ with its image in $[X, \Omega P^{2n+1}(p^r)]$ under the group monomorphism

$$\Omega\tilde{\pi}_* \colon [X, \Omega P(n, p^r)] \hookrightarrow [X, \Omega P^{2n+1}(p^r)].$$

Let $\alpha_k = x_1(P) \cdots x_k(P) \in K_k(P)$. By Corollary 2.8 and Lemma 4.2, we have

$$\alpha_k^{p^{r+1}} \in \mathcal{B}_k(P)$$

for each k. Since $\alpha_k^{p^{r+1}} \in \mathcal{H}_k(P^{2n}(p^r), \Omega P(n, p^r))$, we have

$$\alpha_k^{p^{r+1}} \in \mathcal{B}_k(P) \cap \mathcal{H}_k(P^{2n}(p^r), \Omega P(n, p^r)).$$

With letting $k \to \infty$, we obtain a map

$$f \colon J(P^{2n}(p^r)) \longrightarrow \Omega P(n, p^r)$$

such that the composite $(\Omega\tilde{\pi}) \circ f$ represents the homotopy class

$$\alpha_\infty^{p^{r+1}} \in \mathcal{H}(P^{2n}(p^r), \Omega P^{2n+1}(p^r)) \cong [J(P^{2n}(p^r), \Omega P^{2n+1}(p^r)]$$

whose geometric interpretation is the power map

$$p^{r+1} \colon J(P^{2n}(p^r)) \simeq \Omega P^{2n+1}(p^r) \to \Omega P^{2n+1}(p^r).$$

Thus there is a homotopy commutative diagram

$$\begin{tikzcd}
 & \Omega P(n, p^r) \arrow[d, "\Omega\tilde{\pi}"] \\
\Omega P^{2n+1}(p^r) \arrow[ur, "f"] \arrow[r, "p^{r+1}"] & \Omega P^{2n+1}(p^r),
\end{tikzcd}$$

and hence the result follows. $\square$

5. Applications to the Anick spaces

Let $E^{2n+1}\{p^r\}$ be the homotopy fibre of the inclusion map $P^{2n+1}\{p^r\} \to S^{2n+1}\{p^r\}$, where $S^{2n+1}\{p^r\}$ is the homotopy fibre of the degree map $[p^r]\colon S^{2n+1} \to S^{2n+1}$. Let $F^{2n+1}\{p^r\}$ be the homotopy fibre of the pinch map $P^{2n+1}(p^r) \to S^{2n+1}$. Then there is a homotopy commutative diagram of fibre sequences

$$\begin{tikzcd}
F^{2n+1}\{p^r\} \arrow[r] & P^{2n+1}(p^r) \arrow[r] & S^{2n+1} \\
E^{2n+1}\{p^r\} \arrow[u] \arrow[r, "\sigma"] & P^{2n+1}(p^r) \arrow[u, equal] \arrow[r, "\phi"] & S^{2n+1}\{p^r\} \arrow[u] \\
\Omega^2 S^{2n+1} \arrow[u, "j"] \arrow[r] & * \arrow[u] \arrow[r] & \Omega S^{2n+1}. \arrow[u]
\end{tikzcd} \tag{5.1}$$

Let W_n be the homotopy theoretic fibre of the double suspension $S^{2n-1} \to \Omega^2 S^{2n+1}$. The space W_n is deloopable and its classifying space BW_n is an H-space [7] with a fibre sequence

$$S^{2n-1} \longrightarrow \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n.$$

By [8, Corollary 3.5], the Gray map ν factors through $E^{2n+1}\{p^r\}$ with a homotopy commutative diagram

$$\begin{tikzcd}
E^{2n+1}\{p^r\} \arrow[r, "\nu^E"] & BW_n \\
\Omega^2 S^{2n+1}. \arrow[u, "j"] \arrow[ur, "\nu"']
\end{tikzcd} \tag{5.2}$$

Let R_0 be the homotopy fibre of $\nu^E\colon E^{2n+1}\{p^r\} \to BW_n$. By [8, Theorem 3.8], there is a homotopy commutative diagram of fibre sequences

$$\begin{tikzcd}
T^{2n-1}_\infty(p^r) \arrow[r] \arrow[d] & \Omega S^{2n+1}\{p^r\} \arrow[r] \arrow[d] & BW_n \arrow[d, equal] \\
R_0 \arrow[r, "\sigma_1"] \arrow[d, "\sigma\circ\sigma_1"] & E^{2n+1}\{p^r\} \arrow[r, "\nu^E"] \arrow[d, "\sigma"] & BW_n \\
P^{2n+1}(p^r) \arrow[r, equal] & P^{2n+1}(p^r), &
\end{tikzcd}$$

where the Anick space $T^{2n-1}_\infty(p^r)$ is denoted as T_{2n-1} in [8]. The left column gives a fibre sequence

$$\Omega R_0 \xrightarrow{\Omega(\sigma\circ\sigma_1)} \Omega P^{2n+1}(p^r) \xrightarrow{\partial} T^{2n-1}_\infty(p^r), \tag{5.3}$$

where ∂ is the connecting map as in [8, Corollary 3.9].

Theorem 5.1. *The composite*

$$\Omega P^{2n+1}(p^r) \xrightarrow{p^r} \Omega P^{2n+1}(p^r) \xrightarrow{\partial} T_\infty^{2n-1}\{p^r\}$$

is null homotopic for $p > 3$, $n > 1$ and $r > 1$.

Proof. The assertion follows by using the same arguments in the proof of Theorem 4.1. Here, we choose the subgroup

$$\mathcal{B}_k(R_0) = K_q(P) \cap \mathrm{Im}(\Omega\sigma \circ \sigma_{1*} \colon [(P^{2n}(p^r))^{\times k}, \Omega R_0] \to [(P^{2n}(p^r))^{\times k}, \Omega P^{2n+1}(p^r)])$$

with the property that $\gamma_2(K_k(P)) \leq \mathcal{B}_k(R_0)$ by using the same arguments in the proof of Lemma 4.2. □

Together with [14, Theorem 1], the map $\partial \colon \Omega P^{2n+1}(p^r) \to T_\infty^{2n-1}\{p^r\}$ has a right homotopy inverse after looping, we have the following.

Corollary 5.2. *The space $\Omega T_\infty^{2n-1}\{p^r\}$ has a multiplicative exponent p^r. In particular, $p^r \cdot \pi_*(T_\infty^{2n-1}\{p^r\}) = 0$.* □

Note. Corollary 5.2 is [14, Theorem 2], where $\Omega T_\infty^{2n-1}\{p^r\}$ was denoted as $D(n, r)$ in [14]. Theorem 5.1 improves [14, Theorem 2] in the sense that the p^r power map of $\Omega P^{2n+1}(p^r)$ already goes trivially to the Anick space up to homotopy before looping.

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